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RELATIVISTIC MODELS OF THE UNIVERSE WITH PRESSURE EQUAL TO ZERO AND TIME-DEPENDENT UNIFORMITY

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SUMMARY

The previous work on the zero-density and approximate models of the universe, published in NASA TN D-2601 and TN D-3047, is summarized. The zero-density model is improved so that it is a useful approximation of the exact model to greater values of the red shift, and regions of applicability of the approximate model are determined. Regions where closed-form solutions of the exact model exist, other than zero-density and zero-cosmical-constant models, are defined and the solutions are given. The results of calculations are presented that indicate, for the data considered, all the models, simplified and exact, appear to fit the data equally well even though there is wide variation in the density and acceleration parameters. In addition, the results of the calculations indicate that density and acceleration parameters should be determined as a pair from observational data. It is concluded that data consistent with that for galaxies at red shifts greater than one-half are required before a model universe can be selected and a simplified model defined.

INTRODUCTION

One of the uses of the theory of model universes is the analysis of observational data to obtain an indication of the structure of the universe and how the universe has evolved. There are several theories of gravitation that provide theories of model universes. The discussion in this paper is limited to models of the universe based on general relativity. The key assumptions made are that the models have time-dependent uniformity and that the pressure is low enough to be assumed negligible when compared with the density term. The exact equations for the relativistic models with zero pressure and time uniformity are derived. The exact equations contain elliptic integrals and therefore cannot, in general, be integrated to obtain analytic solutions. The use of the simplified models, presented in references 1 and 2, is discussed and additional material relative to these models and their use is included. (Appendix A gives the regions of usefulness of the approximate model.) In addition to the zero-density and approximate models and those models in which the cosmical constant is zero, there are other models that have solutions in terms of simple functions. McVittie (ref. 3) has discussed these models from the point of view of special solutions of the elliptic integrals. In this paper, an algebraic approach is used that permits the establishment of the combinations of the density and acceleration parameters that give additional simple function solutions.

The differences between the meaningful simplified models and the exact models are discussed, and the results of calculations with the various models to determine the density and acceleration parameters are compared. Appendix B lists the symbols used in the analysis.

RELATIVISTIC MODEL UNIVERSES

The space-time metric for relativistic models of the universe that have time-dependent uniformity is

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2} \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2}{\left(1 + \frac{kr^2}{4}\right)^2} \tag{1}$$

where ds has the dimensions of time along the line element; c is the speed of light in vacuo; t is time; r, θ, and ϕ are the dimensionless coordinates of a point in the metric subspace; k is the curvature constant; and R is the scale factor that describes the geometrical history of the universe and has the dimensions of length. The relation between the space time of experience, the metric space, and the physical content of space time is provided by the field equations of general relativity:

$$E_{\mu\nu} - \frac{1}{2}\left(Gg_{\mu\nu} - 2\Lambda g_{\mu\nu}\right) = -\kappa c^2 T_{\mu\nu} \tag{2}$$

where $E_{\mu\nu}$ is the contracted Riemann-Christoffel tensor; E, the curvature scalar; $g_{\mu\nu}$, the metrical tensor; Λ, the cosmical constant; κ is a constant $= \frac{8\pi G}{c^2}$, and G is the universal constant of gravitation. The material energy tensor $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)V_\mu V_\nu - \frac{p}{c^2} g_{\mu\nu} \tag{3}$$

for a single stream isotropic fluid. In equation (3), ρ and p are the density and pressure, respectively, and $V_\mu = \frac{dx^\mu}{ds}$. The subscripts μ and ν appearing in equations (2) and (3) are the usual tensor indices and can assume values 0 to 3.

The field equations (eq. (2)) are written with the cosmical constant. Einstein's original formulation was

$$E_{\mu\nu} - \frac{1}{2} E g_{\mu\nu} = -\kappa c^2 T_{\mu\nu} \tag{4}$$

The cosmical constant was introduced by Einstein in reference 4 in order to obtain a stationary model universe which at that time was indicated by observation. After observation had established the expansion of the universe and it had been shown that expanding universe solutions existed for $\Lambda = 0$ as well as $\Lambda \neq 0$ (ref. 5), Einstein discontinued the use of the cosmical constant on the grounds of logical economy (ref. 6) and maintained this view through his last discussion of the question (ref. 7). The question of the cosmical constant is not as simple as the foregoing discussion would indicate, and Einstein himself was not consistent in his use or rejection of the cosmical constant. When working with the electromagnetic field (ref. 8), Einstein found and used the cosmical constant as "a constant of integration," whereas when working with the ponderable matter field, he regarded it as a constant peculiar to the theory. Weyl (ref. 9) presents a rigorous mathematical derivation of the field equations and shows that the cosmical constant is a necessary term in the field equations. McVittie (ref. 10) and Heckmann (ref. 11) both support the retention of the cosmical constant on mathematical grounds. Because the mathematical evidence indicates that the cosmical constant is a necessary term in the field equations and the rejection of this term seriously limits the generality of the theory of relativity, the cosmical constant is retained in this study.

The substitution of equations (1) and (3) into equation (2) and carrying out the indicated operations gives the well-known form of the Einstein field equations for an isotropic homogeneous universe

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \kappa p = -\frac{kc^2}{R^2} + \Lambda \tag{5a}$$

$$\frac{\dot{R}^2}{R^2} - \frac{\kappa c^2 \rho}{3} = -\frac{kc^2}{R^2} + \frac{\Lambda}{3} \tag{5b}$$

where R is the scale factor of the universe and the dots over the symbols indicate differentiation with respect to time. The solutions $R(t)$ of these differential equations describe the geometrical history of the universe. There are many possible solutions of equations (5), but it is required that the solutions have physical significance. Since observation indicates the presence of matter and radiation in the universe, only those sets of k, Λ, and R that give $\rho \geqq 0$ and $p \geqq 0$ are of interest.

An examination of equations (5) shows that Λ has the dimensions of $\sec^{-2}$. The cosmical constant can be greater than, less than, or equal to zero. When Λ is less than zero, it represents an acceleration term that acts in conjunction with gravity forces to retard expansion. When Λ is greater than 0, it represents an acceleration that opposes that of gravity in the retardation of the expansion of the universe.

Like Λ, k can be greater than, less than, or equal to zero. The sign of k determines the type of space that is being considered. For $k > 0$, the space is spherical; for $k < 0$, the space is hyperbolic; and for $k = 0$, the space is Euclidean. In the derivation of equations (5), k was adjusted so that it can be considered a space-curvature constant that can assume a value of 1, -1, or 0.

In equations (5) p and ρ are functions of time; thus, equations (5) are not determinate since three unknowns must be determined from two equations. Solutions to equations (5) can be obtained only if some assumption is made with regard to p and ρ. Before such an assumption is made, however, it is of interest to look at the magnitudes of the density and pressure terms. The pressure is composed of two terms, one due to matter and the other due to radiation. In a low-density fluid the pressure due to material density is given by two-thirds of the energy density due to random motions which reduces to $\rho_o v_o^2$ where v_o is the random radial velocity. The expression for total pressure can now be written as $p = \rho_o v_o^2 + \frac{aT^4}{3}$ where ρ_o is the material density, $a = 7.564 \times 10^{-15}$, and T is the radiation temperature in degrees Kelvin. The term $aT^4/3$ is the radiation pressure. A good value for ρ_o, based on observation, is 10^{-30} g/cm^3. (See refs. 12 and 13.) The random radial velocity has not been accurately determined but is probably less than 3×10^7 cm/sec (ref. 14). These values of ρ_o and v_o give a pressure due to material density on the order of 10^{-15}. The radiation temperature of intergalactic space, for the present epoch, is about 3° K (refs. 15 and 16) and corresponds to a radiation pressure on the order of 10^{-13}. These results indicate that for the present epoch, radiation pressure dominates the pressure term.

Density is also composed of two terms, the material density (that is, the density due to matter) and the density equivalent of radiation $\left(\text{that is, } aT^4/c^2\right)$. For a radiation temperature of 3° K the density equivalent of radiation is on the order of 10^{-33} and thus smaller by a factor of 10^3 than the density due to matter.

From equations (5) the relation between pressure and density is

$$\frac{d}{dt}\left(\rho R^3\right) + \frac{p}{c^2}\frac{d}{dt}R^3 = 0 \tag{6}$$

This equation can be written as $R^3 \frac{d\rho}{dt} + \left(\rho + \frac{p}{c^2}\right)\frac{dR^3}{dt} = 0$. The results of the analysis of density and pressure for the present epoch, when applied to the coefficient of dR^3/dt, indicate that p/c^2 will be less than ρ by a factor of 10^3; thus, pressure can be assumed to make a negligible contribution to the value of dR^3/dt when compared with density. Thus, in the models of the universe discussed herein, the pressure is assumed to be zero. As a consequence of the assumption of zero pressure, integration of equation (6) gives ρR^3 = Constant = $\rho_o R_o{}^3$ and gives density as function of R and R_o, $\rho = \left(\frac{R_o}{R}\right)^3 \rho_o$. The subscript o indicates present values of ρ and R.

Under the assumption of zero pressure, equations (5) become

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = -\frac{kc^2}{R^2} + \Lambda \tag{7a}$$

$$\frac{\dot{R}^2}{R^2} - \frac{kc^2\rho}{3} = -\frac{kc^2}{R^2} + \frac{\Lambda}{3} \tag{7b}$$

Equations (7) are the equations that are used in the development of model universes. When equations (7) are written for the present epoch and the following definitions are substituted

$$\frac{\dot{R}_o}{R_o} = H_o$$

$$\frac{\ddot{R}_o}{R_o} = -q_o H_o{}^2$$

$$\sigma_o = \frac{4\pi G\rho_o}{3H_o{}^2}$$

it is found that

$$\Lambda = 3H_o{}^2(\sigma_o - q_o) \tag{8}$$

$$\frac{kc^2}{R_o{}^2} = H_o{}^2(3\sigma_o - q_o - 1) \tag{9}$$

where the density parameter σ_o and acceleration parameter q_o are pure numbers and the Hubble parameter H_o has dimensions of $\sec^{-1}$. If σ_o, q_o, and H_o can be determined from observation, equations (8) and (9) can be used to determine the type of space curvature and the value of the cosmical constant. If the curvature constant and cosmical constant are known, the type of model universe can be determined. For zero-pressure models, the models that correspond to $k = \pm 1$ or 0 for $\Lambda \lesseqgtr 0$ can be determined from equation (7b) without reference to σ_o and q_o.

To determine R(t) for the various combinations of k and Λ, equation (7b) is solved for time; the solution yields

$$t_o - t = \int_R^{R_o} \frac{R^{1/2} dR}{\sqrt{\frac{\Lambda}{3} R^3 - kc^2 R + 2H_o^2 \sigma_o R_o^3}} \tag{10}$$

and the inversion of the time solution gives $R(t)$. If the integration indicated in equation (10) is carried out, the solution is in terms of elliptic functions. However, study of the cubic in equation (10) provides information on the behavior of R in different models. This type of analysis has been made by both Robertson (ref. 17) and Bondi (ref. 18), and table I summarizes the results of these studies for model universes in which $\rho > 0$ and $p = 0$.

The material presented in table I is self-explanatory for $k = -1$ and 0 and $k = 1$, $\Lambda \leqq 0$ since for a given combination of k and Λ, only one type of model universe is possible. When $k = 1$ and $\Lambda > 0$, six different models of the universe can occur. The specific model depends on whether Λ is greater than, equal to, or less than a specific value called the critical value (designated Λ_e). If the cubic in equation (10) is set equal to zero and solved for Λ, it is found that

$$\Lambda = \frac{3kc^2}{R^2} - \frac{6H_o^2 R_o^3 \sigma_o}{R^3} \tag{11}$$

and the critical value is that value of Λ for which $d\Lambda/dR = 0$ is given by

$$\Lambda_e = \frac{k^3 c^6}{9H_o^4 R_o^6 \sigma_o^2} \tag{12}$$

The value of R associated with Λ_e called R_e is given by

$$R_e = \frac{3H_o^2 R_o^3 \sigma_o}{kc^2} \tag{13}$$

If $\Lambda > \Lambda_e > 0$, the model is Le Maitre's model (indicated as model 9 in table I). When $\Lambda = \Lambda_e$, three different models are possible. The model in this case depends on the relationship of R_o to R_e. When $R_o < R_e$ (model 12), a model that expands asymptotically from $R = 0$ to $R = R_e$ occurs; when $R_o = R_e$, the Einstein static model (model 10) is indicated; and when $R_o > R_e$, the Eddington-LeMaitre model (model 11) is the model that would represent the universe. When $\Lambda < \Lambda_e$, the cubic has two roots, R_1 and R_2, $R_1 < R_2$; when $R_o < R_1$, then model 13 occurs; and when $R_2 \leqq R_o$, the model is model 14.

Some interesting points about the models of the universe are shown in table I. In most cases a singular point at $R = t = 0$ indicates that expansion started from a zero radius. This singular point is purely mathematical and the theory of model universes does not provide information concerning the physical state of the universe. There is

TABLE I.- SUMMARY OF MODEL UNIVERSES

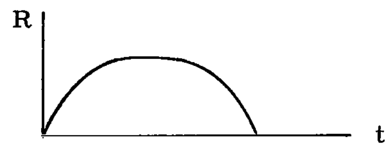
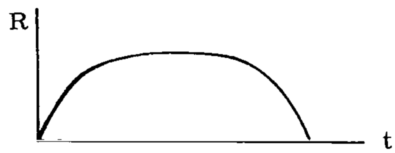
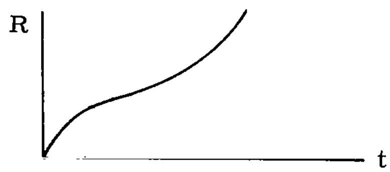
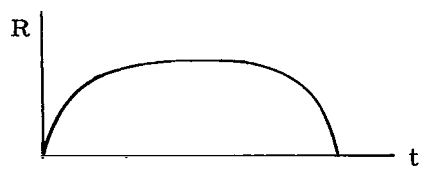
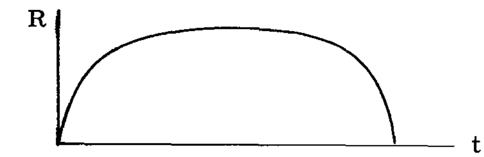
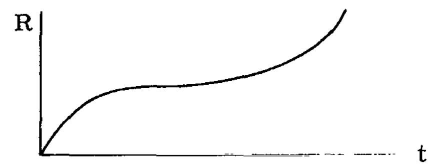
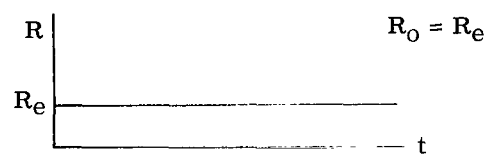
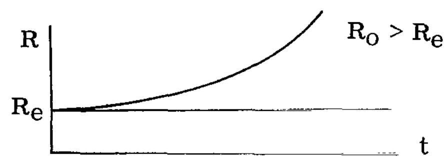
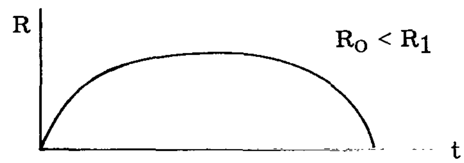
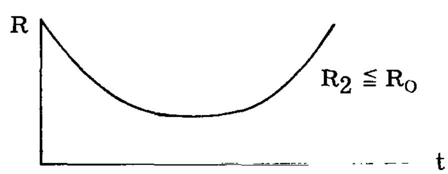
	Type	R(t)	Model
		$k = -1$	
$\Lambda < 0$	Oscillating		1
$\Lambda = 0$	Expanding		2
$\Lambda > 0$	Expanding		3
		$k = 0$	
$\Lambda < 0$	Oscillating		4
$\Lambda = 0$	Expanding		5
$\Lambda > 0$	Expanding		6
		$k = 1$	
$\Lambda < 0$	Oscillating		7

TABLE I.- SUMMARY OF MODEL UNIVERSES – Concluded

	Type	R(t)	Model
		$k = 1$	
$\Lambda = 0$	Oscillating		8
$\Lambda > \Lambda_e > 0$	Expanding Le Maitre's model		9
$\Lambda = \Lambda_e$	Stationary Einstein's model		10
$\Lambda = \Lambda_e$	Expanding Eddington-Le Maitre model		11
$\Lambda = \Lambda_e$	Expanding		12
$\Lambda_e > \Lambda > 0$	Oscillating		13
$\Lambda_e > \Lambda > 0$	Oscillating		14

some indication (ref. 3) that the singularity may have been introduced by the assumption of zero pressure, and if a finite pressure model is considered, $R > 0$ at $t = 0$. The other point of interest concerns the models of the universe in which $\Lambda = 0$. When $\Lambda = 0$, equation (10) can be integrated by simple functions and the properties of the $\Lambda = 0$ models are given by simple analytical expressions. However, by equation (8) σ_o must be equal to q_o where $\Lambda = 0$. This condition is most restrictive; in addition to requiring the equality of σ_o and q_o, the universe cannot be accelerating. Acceleration requires negative q_o and negative q_o gives negative density; the $\Lambda = 0$ models and negative density cannot exist in the physical universe. The mathematical simplicity of $\Lambda = 0$ is most attractive; however, the restrictive nature of the assumption lends support to the mathematical reasons previously cited for retaining Λ in the field equations. If Λ is actually zero, the analysis of observational data will show this condition and firm justification will exist for setting $\Lambda = 0$.

Connection Between Theory and Observation

The classes of observational data that can be used to determine a model of the universe are apparent magnitude, red shift, angular diameter, and number counts. The isophotal diameter is not considered because of difficulties associated with its use. (See ref. 14.) In order to determine a model of the universe from observational data, it is necessary to write down an equation that expresses a relation of two or more of the classes of observational data as a function of σ_o and q_o. From this equation, expressions relating apparent magnitude and red shift, angular diameter and red shift, and number counts and red shift have resulted. These relationships, in their most general form, are:

Between red shift and apparent magnitude (refs. 1 and 3):

$$m - K = 5 \log R_o(1 + z)S(\omega) + M_o + \Delta M_o - 5 \tag{14}$$

Between angular diameter and red shift (ref. 19):

$$\phi = \frac{\zeta(1 + z)}{R_o S(\omega)} \tag{15}$$

Between number of galaxies and red shift (refs. 19 and 20):

$$N(m) = \frac{2\pi n R_o^3}{(-k)Q}\left(\sinh \sqrt{-k}\omega \cosh \sqrt{-k}\omega - \sqrt{-k}\omega\right) \tag{16}$$

where

m apparent magnitude

K red-shift correction

z red shift

M_o absolute magnitude of equivalent local source

ΔM_o evolutionary correction to M_o

ϕ angular diameter

ζ linear diameter

$N(m)$ number of like sources brighter than a given magnitude

n number of like sources per unit volume

Q number of square degrees in the celestial sphere

Equation (14) is the red-shift—magnitude relation and the argument of the log term $R_o(1 + z)S(\omega)$ is the luminosity distance D_l, the distance that light travels from the source to the observer. Equation (15) is the angular-diameter—red-shift relation and the denominator of the right-hand side $R_oS(\omega)$ is the distance by measuring rod D_c. Equation (16) is the number count relation and, as indicated in reference 21, it is not too useful with presently available data.

An inspection of equations (14), (15), and (16) shows that R_o, ω, and $S(\omega)$ are functions of the radial metric variable r of equation (1), and they need to be expressed as functions of σ_o, q_o, and z in order to make the equations useful for the analysis of observational data. Equation (9) gives R_o as a function of σ_o and q_o. The variable ω and the function $S(\omega)$ can be obtained from equations (1) and (7) as functions of σ_o, q_o, and z.

Equation (1) is first rewritten with the aid of the transformation

$$\xi = \frac{R_o}{\sqrt{-k}} \sinh \sqrt{-k}\omega = \frac{R_o r}{1 + \frac{kr^2}{4}} \tag{17}$$

so that the metric (eq. (1)) becomes

$$ds^2 = dt^2 - \frac{R^2}{c^2 R_o^2}\left[\frac{d\xi^2}{1 - \frac{k\xi^2}{R_o^2}} + \xi^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right] \tag{18}$$

If the coordinates are chosen so that the origin is at the observer, by spherical symmetry θ and ϕ are constants for any given light ray. The coordinates are so adjusted that

$\theta = \phi = 0$. Light travels along a null geodesic and $ds^2 = 0$; thus, equation (18) can now be written as

$$c \frac{dt}{R} = \frac{d\xi}{R_o\sqrt{1 - \frac{k\xi^2}{R_o^2}}} \tag{19}$$

Differentiation of equation (17) gives

$$d\omega = \frac{d\xi}{R_o\sqrt{1 - \frac{k\xi^2}{R_o^2}}}$$

If this definition of $d\omega$ is used, equation (19) can now be written as

$$d\omega = c \frac{dt}{R} = \frac{d\xi}{R_o\sqrt{1 - \frac{k\xi^2}{R_o^2}}}$$

and integrating yields

$$\omega = \frac{1}{\sqrt{-k}} \sinh^{-1} \sqrt{-k} \frac{\xi}{R_o} = c \int_t^{t_o} \frac{dt}{R} \tag{20}$$

From equation (20),

$$S(\omega) = \frac{1}{\sqrt{-k}} \sinh \sqrt{-k}\omega \tag{21a}$$

and

$$\omega = c \int_t^{t_o} \frac{dt}{R} = \int_R^{R_o} \frac{dR}{\dot{R}R} \tag{21b}$$

Through the use of equation (7b), the definition of σ_o, and the relation between ρ and ρ_o for zero pressure models $\left(\rho_o = \rho \frac{R^3}{R_o^3}\right)$, equation (21b) can be written as

$$\omega = \int_R^{R_o} \frac{dR}{R^{1/2}\sqrt{2H_o^2 R_o^3 \sigma_o - kcR^2 + \frac{\Lambda}{3} R^3}} \tag{22}$$

Equation (22) can be expressed in the terms of σ_o, q_o, and the red shift z through the use of equations (8) and (9) and the relation $R_o/R = 1 + z$. Equation (22) can now be written

$$\omega = \frac{c}{H_o R_o} \int_0^z \frac{dz}{\left[2\sigma_o z^3 + \left(3\sigma_o + q_o + 1\right)z^2 + 2\left(q_o + 1\right)z + 1\right]^{1/2}} \tag{23}$$

Equations (14) to (16) can now be expressed in terms of σ_o, q_o, H_o, and z by the use of equations (9), (21a), and (23).

Integrations To Obtain ω

Equation (23) is an elliptic integral and cannot, in general, be integrated in terms of simple functions. There are certain assumptions which, when made, permit the integration of equation (23) in terms of simple functions. The first is to assume that the cubic has two equal roots; the denominator then reduces to $(z - b)\sqrt{z - a}$ and can be integrated. The two equal roots can occur only for specific values of σ_o and q_o, and for this reason, is of very limited usefulness. However, one of the cases of two equal roots gives the solutions for the $\Lambda = 0$ models. A second approach is to assume that density is zero. The assumption of zero density reduces the cubic to a quadratic which can be integrated directly. The solutions for the zero-density models are exact from a mathematical point of view, but, when used, it is assumed that the density of the universe is so small that the physical situation existing in the universe can be approximated by the zero-density model. The alternative approach to those already suggested seeks to obtain a mathematical approximation for equation (23). Such an approximation occurs when the term $2\sigma_o z^3$ in the cubic can be neglected with respect to the other terms in the equation.

Methods for integrating equation (23).- If equation (23) is written in the form

$$\omega = \frac{c}{H_o R_o} \int_0^z \frac{dz}{\sqrt{(z - \alpha)(z - \beta)(z - \eta)}}$$

it can be integrated by the methods given in standard texts on elliptic integrals. Such solutions, in general, provide only numerical answers once the roots are known, and thus imply that σ_o and q_o are known. The alternative is to integrate equation (23) by numerical methods on a digital computer. In the problem of analysis of observational data, σ_o and q_o are the unknown quantities, and expressions for ω of the form $\omega = f(\sigma_o, q_o, z)$ are more useful.

Two equal roots.- There are two separate cases when there are two equal roots. In the first, $\Lambda \neq 0$ and in the second, $\Lambda = 0$. In the first case $\Lambda \neq 0$, the variable ω is given by

$$\omega = \frac{c}{H_o R_o \sqrt{2\sigma_o}} \int_0^z \frac{dz}{\left(z + \frac{q_o + 1}{3\sigma_o}\right)\sqrt{z + \frac{9\sigma_o - (q_o + 1)}{6\sigma_o}}} \tag{24}$$

which for $k = 1$ integrates to

$$\omega = \log \frac{\sqrt{6\sigma_0 z + 9\sigma_0 - (q_0 + 1)} - \sqrt{3}\sqrt{3\sigma_0 - (q_0 + 1)}}{\sqrt{6\sigma_0 z + 9\sigma_0 - (q_0 + 1)} + \sqrt{3}\sqrt{3\sigma_0 - (q_0 + 1)}} - \frac{\sqrt{9\sigma_0 - (q_0 + 1)} + \sqrt{3}\sqrt{3\sigma_0 - (q_0 + 1)}}{\sqrt{9\sigma_0 - (q_0 + 1)} - \sqrt{3}\sqrt{3\sigma_0 - (q_0 + 1)}} \tag{25}$$

and for $k = -1$

$$\omega = 2\left[\tan^{-1} \frac{\sqrt{6\sigma_0 z + 9\sigma_0 - (q_0 + 1)}}{\sqrt{3}\sqrt{q_0 + 1 - 3\sigma_0}} - \tan^{-1} \frac{\sqrt{9\sigma_0 - (q_0 + 1)}}{\sqrt{3}\sqrt{q_0 + 1 - 3\sigma_0}}\right] \tag{26}$$

The necessary condition for these solutions to exist is given by

$$3\sigma_0 - (q_0 + 1)^3 - 27\sigma_0^2(\sigma_0 - q_0) = 0 \tag{27}$$

These pairs of σ_0 and q_0 that satisfy equation (27) are shown in figure 1 for $-4.0 \leqq q_0 \leqq 6.0$ and $0 \leqq \sigma_0 \leqq 2.0$. The solutions for ω correspond to models with $k = \pm 1$. The solutions for $k = 0$ models are points and occur when $\sigma_0 = q_0 = 0.5$ and $\sigma_0 = 0$, $q_0 = -1.0$. Equations (25) and (26) do not apply at these points and special expressions are required for ω. One of these points $\sigma_0 = q_0 = 0.5$ belongs with the $\Lambda = 0$ models and is discussed later. The second point $\sigma_0 = 0$, $q_0 = -1.0$ is on the curve for negative q_0 in figure 1. The ω for this point is given by

$$\omega = \frac{c}{H_0 R_0} z \tag{28}$$

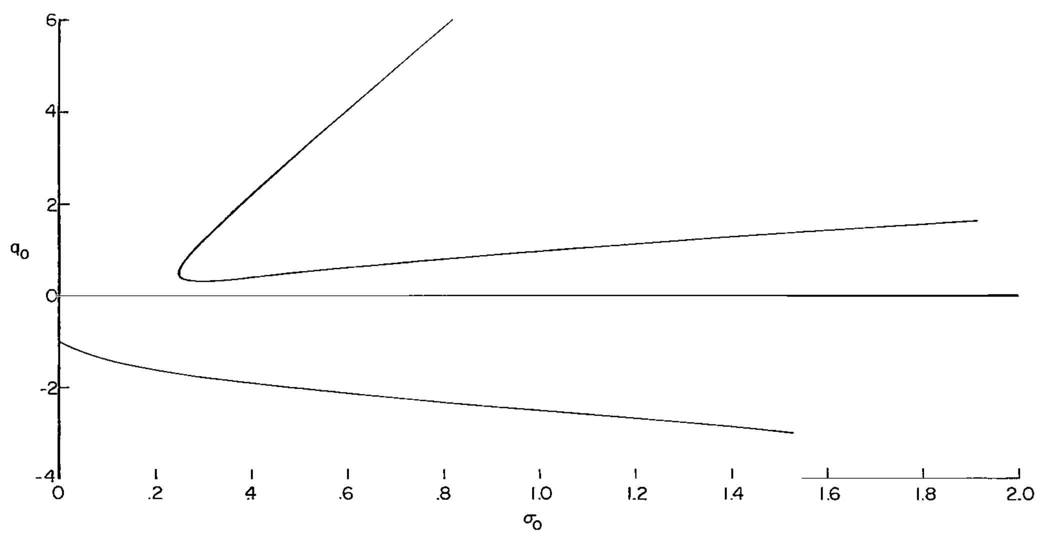


Figure 1.- Combinations of the density and acceleration parameters, σ_0 and q_0, that satisfy the conditions for the integration of equation (27).

In the second case $\Lambda = 0$, where equal roots exist, ω is given by

$$\omega = \frac{c}{H_o R_o \sqrt{2\sigma_o}} \int_0^z \frac{dz}{(z + 1)\sqrt{z + \frac{q_o + 1 - \sigma_o}{2\sigma_o}}} \tag{29}$$

and the necessary condition for the solutions to be valid is that $\sigma_o = q_o$; by equation (8) this relationship corresponds to $\Lambda = 0$. In the case of $\Lambda = 0$ models, ω is obtained in a more usable form if Λ is set to zero in equation (22) and then integrated as was done by Mattig (ref. 22). The resulting ω equations are for $k = 1$,

$$\omega = \cos^{-1}\left[\frac{c^2}{H_o^2 R_o^2 \sigma_o (1 + z)} - 1\right] - \cos^{-1}\left(\frac{c^2}{H_o^2 R_o^2 \sigma_o} - 1\right) \tag{30}$$

for $k = 0$,

$$\omega = \sqrt{\frac{2c^2}{H_o^2 R_o^2 \sigma_o}} - \sqrt{\frac{2c^2}{H_o^2 R_o^2 \sigma(1 + z)}} \tag{31}$$

and, for $k = -1$,

$$\omega = \cosh^{-1}\left(\frac{c^2}{H_o^2 R_o^2 \sigma_o} + 1\right) - \cosh^{-1}\left(\frac{c^2}{H_o^2 R_o^2 \sigma_o (1 + z)} + 1\right) \tag{32}$$

When $\Lambda = 0$, $k = 0$ only when $q_o = 0.5$. In $\Lambda = 0$ models, q_o must be positive or equal to zero since negative q_o would give negative density (that is, $\sigma_o < 0$); and since objects (galaxies, quasi-stellar radio sources) exist in the universe, the density and consequently σ_o cannot be negative. In both cases of two equal roots $\Lambda \neq 0$ and $\Lambda = 0$, equation (9) is used to determine k and, consequently, determine the form of ω to be used.

Equations (25), (26), (28), and (30) to (32) represent exact solutions for ω and impose a severe restraint on the relation between σ_o and q_o. In both cases, $\sigma_o \neq q_o$ and $\sigma_o = q_o$, it is necessary for σ_o and q_o to have a relationship that comes solely from the mathematics of the problem and is not a result of observation or the physics of the problem.

The zero-density model.- Observational results (ref. 16) indicate that the density of the universe is probably close to 10^{-30} g/cm^3. This level of density means that σ_o is about 0.05, and for $z < 1.0$ the terms involving σ_o will contribute little to the cubic in equation (23). It is therefore interesting to investigate the zero-density models, which are exact mathematical models, as approximations for a very low-density universe. When σ_o is set equal to zero, equation (23) becomes

$$\omega = \frac{c}{H_0 R_0} \int_0^z \frac{dz}{\left[-\frac{kc^2}{H_0^2 R_0^2}(1 + z)^2 - q_0 \right]^{1/2}} \tag{33}$$

Figure 2 compares the quantity $\omega H_0 R_0/c$ as calculated by equations (23) and (33). The use of $\omega H_0 R_0/c$ permits the effect of the zero-density assumption on the integral to be determined. The differences between the exact and zero-density models are small to $z = 1.0$ for $q_0 = 0.5$ and decrease with increasing q_0. For $z > 1.0$, the differences increase. This increase occurs because the terms involving density are z^2 and z^3 and the contribution of these terms to the cubic in equation (23) builds up very quickly above $z = 1.0$. The good agreement obtained between $\omega H_0 R_0/c$ for the exact and zero-density solutions indicates that for low density, the zero-density model could be used in place of the exact model. When a simple model is used to replace a more complicated exact model, it is always good to have some idea of the differences between the simple and exact model. Systematic calculations with the exact and zero-density solutions showed that if the following inequality was satisfied

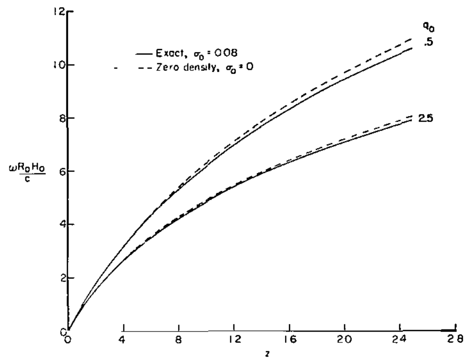


Figure 2.- Differences in $\frac{\omega R_0 H_0}{c}$ for the exact finite density and zero-density models of the universe. (Eqs. (23) and (33).)

$$\sigma_0 \leqq 0.075 q_0 (q_0 - 1) + 0.1 \tag{34}$$

the difference in $\omega H_0 R_0/c$ for the exact and zero-density solutions would be 2 percent or less at $z = 1.0$.

Integration of equation (33) gives a general solution for ω which is

$$\omega = \frac{1}{\sqrt{-k}} \log \frac{\frac{\sqrt{-k}c}{H_0 R_0}(1 + z) + \sqrt{\frac{-kc^2}{H_0^2 R_0^2}(1 + z)^2 - q_0}}{\frac{\sqrt{-k}c}{H_0 R_0} + \sqrt{\frac{-kc^2}{H_0^2 R_0^2} - q_0}} \tag{35}$$

and $S(\omega)$ is given by

$$S(\omega) = \frac{c}{H_0 R_0 q_0} \left[\sqrt{(q_0 + 1)(1 + z)^2 - q_0} - (1 + z) \right] \tag{36}$$

As shown in figures 3 and 4, the agreement between the exact and zero-density solutions for ω and $S(\omega)$ is poor compared with the agreement for the corresponding solutions for $\omega H_0 R_0/c$.

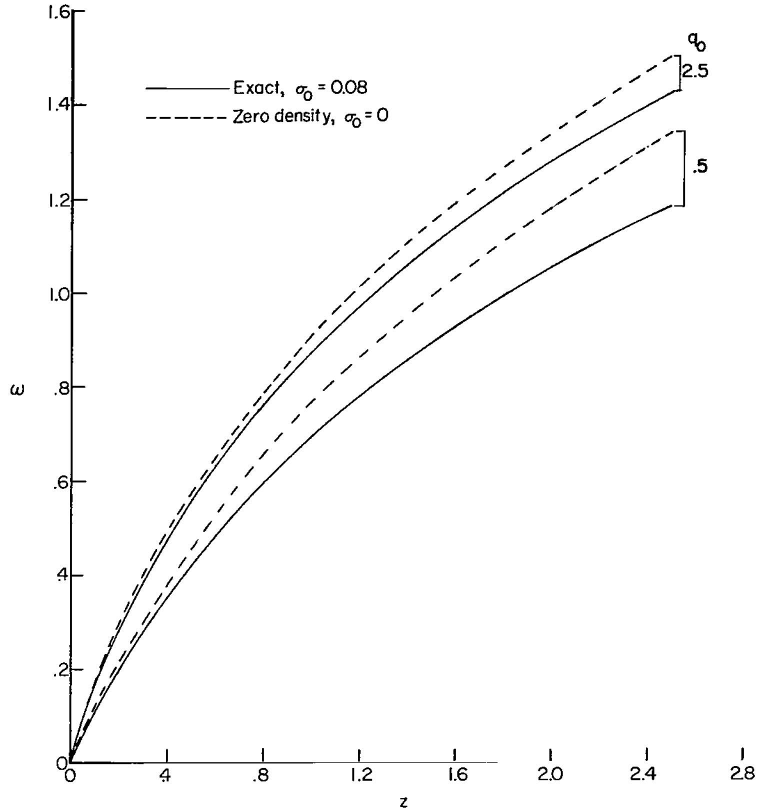


Figure 3.- Differences in ω for the exact finite density and zero-density models of the universe. (Eqs. (23) and (35).)

Both ω and $S(\omega)$ (see eqs. (23), (35), and (36)) are multiplied by a coefficient that contains $1/R_0$. In reference 1 it was shown that the assumption of zero density introduced large differences in R_0. In addition to the differences in the integral, which were found to be small (see fig. 2), the only term that can introduce additional differences, such as those shown in figures 3 and 4 as compared with figure 2, is R_0. This relation suggests that if the zero-density result is multiplied by R_0 for the zero-density model and divided by R_0 for the exact model, the differences between the two models should approach the differences shown in figure 2. Accordingly, ω^* and $S^*(\omega)$ to replace ω and $S(\omega)$, respectively, of the zero-density model are defined as follows:

$$\omega^* = \omega_d \frac{(R_0)_d}{(R_0)_e} \tag{37}$$

$$S^*(\omega) = \left[S(\omega)\right]_d \frac{(R_0)_d}{(R_0)_e} \tag{38}$$

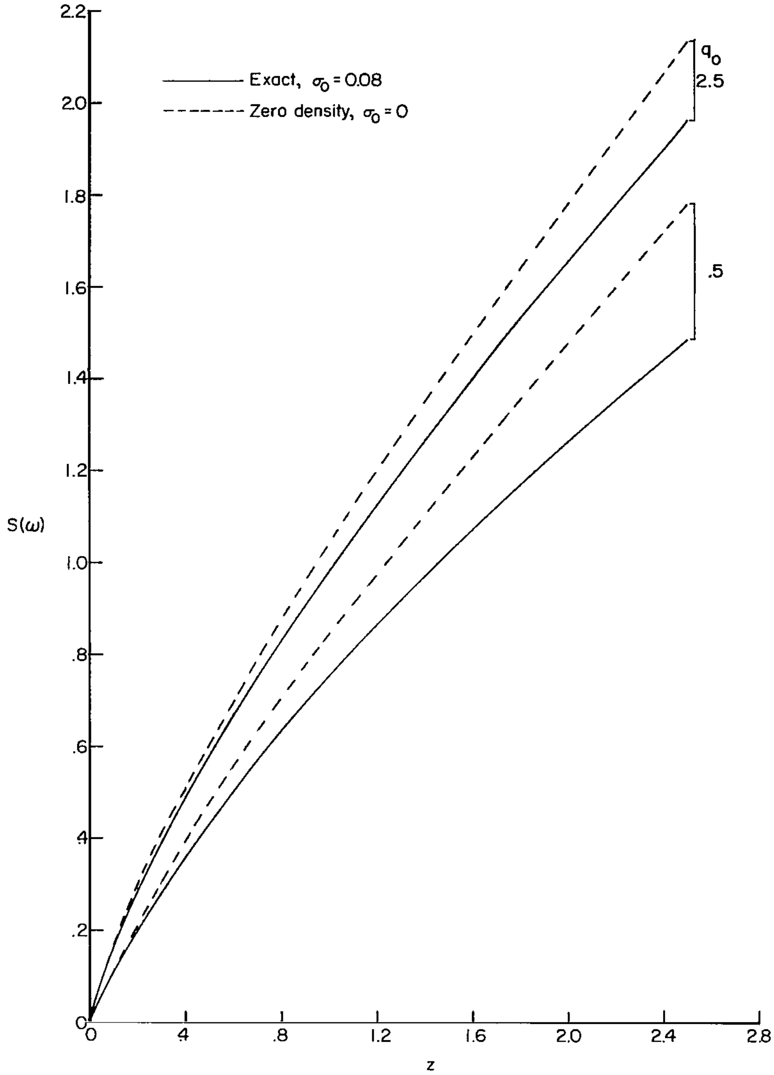


Figure 4.- Differences in $S(\omega)$ for the exact finite density and zero-density models of the universe. (Eqs. (21a), (23), and (36).)

where the subscripts d and e stand for the zero density and exact models, respectively. As shown in figures 5 and 6, the differences between ω^* and ω and between $S^*(\omega)$ and $S(\omega)$, where ω and $S(\omega)$ are for the exact model, approach the differences found for $\omega H_0 R_0/c$.

The use of ω^* and $S^*(\omega)$ is required only when it is desired to investigate the variables by themselves or when ω and $S(\omega)$ are substituted in an equation that does not cancel the $1/R_0$ term that is introducing the large differences. Therefore, in equations (14) and (15), $S^*(\omega)$ should not be used, whereas in equation (16) ω^* and $S^*(\omega)$

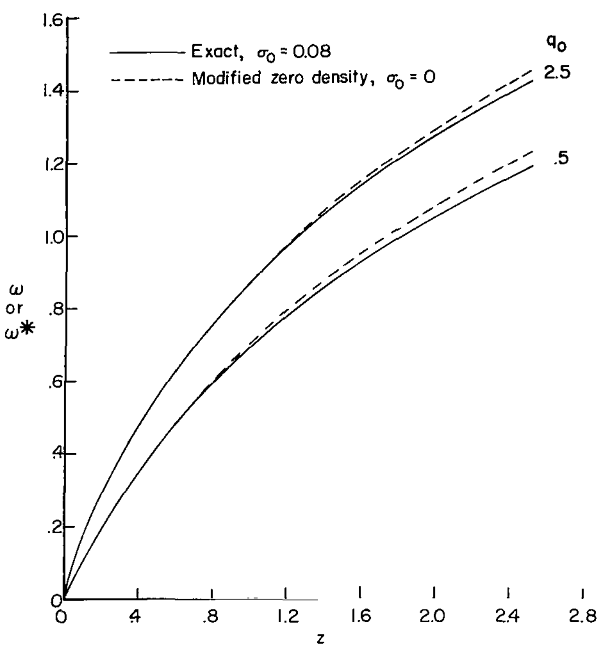


Figure 5.- Differences in ω or ω^* for the exact finite density and modified zero-density models of the universe. (Eqs. (23) and (37).)

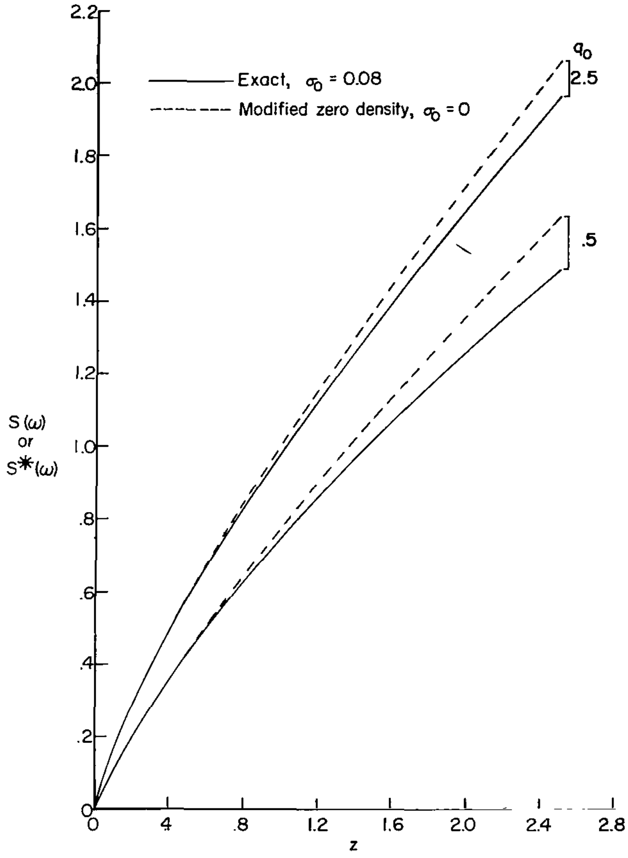


Figure 6.- Differences in $S(\omega)$ or $S^*(\omega)$ for the exact finite density and modified zero-density models of the universe. (Eqs. (21a), (23), and (38).)

should be used. Lastly, it should be noted that the use of ω^* and $S^*(\omega)$ requires that an estimate of the average density of the universe be available.

If the zero-density model is to be used for the analysis of observational data, the k and Λ associated with the zero-density model must have the same signs as the k and Λ of the nonzero-density model as long as $\sigma_0 = q_0$. This requirement means that zero-density models cannot be used for the analysis of observational data unless the following conditions in the observed universe are present: For $q_0 > 0$, the zero-density model can be used if $q_0 > \sigma_0$ and $q_0 + 1 > 3\sigma_0$; for $-1 < q_0 < 0$, if $1 > 3\sigma_0 - q_0$; for $q_0 < -1$, if there is no restriction on q_0.

Approximate model solutions for ω. The results shown in figure 3 for $\omega H_0 R_0/c$ indicate that the integral in equation (23) is very little affected by σ_0. Therefore, an investigation was made of the effect of assuming that for small σ_0 and $z \leqq 1$, the term $2\sigma_0 z^3$ makes a negligible contribution to the cubic in equation (23). This assumption is attractive because the only differences introduced are in the integral $\omega H_0 R_0/c$, because R_0 is not affected, and because the resulting solutions permit the determination of σ_0

from observational data. When it is assumed that $2\sigma_0 z^3$ is a term that makes a negligible contribution to the cubic in equation (23), equation (23) becomes

$$\omega = \frac{c}{H_0 R_0} \int_0^z \frac{dz}{\sqrt{\left(3\sigma_0 + q_0 + 1\right)z^2 + 2\left(q_0 + 1\right)z + 1}} \tag{39}$$

Figure 7 compares the $\omega H_0 R_0/c$ obtained from equation (23) and from equation (39). As can be seen, the difference between the solutions is practically nonexistent out to $z = 1$.

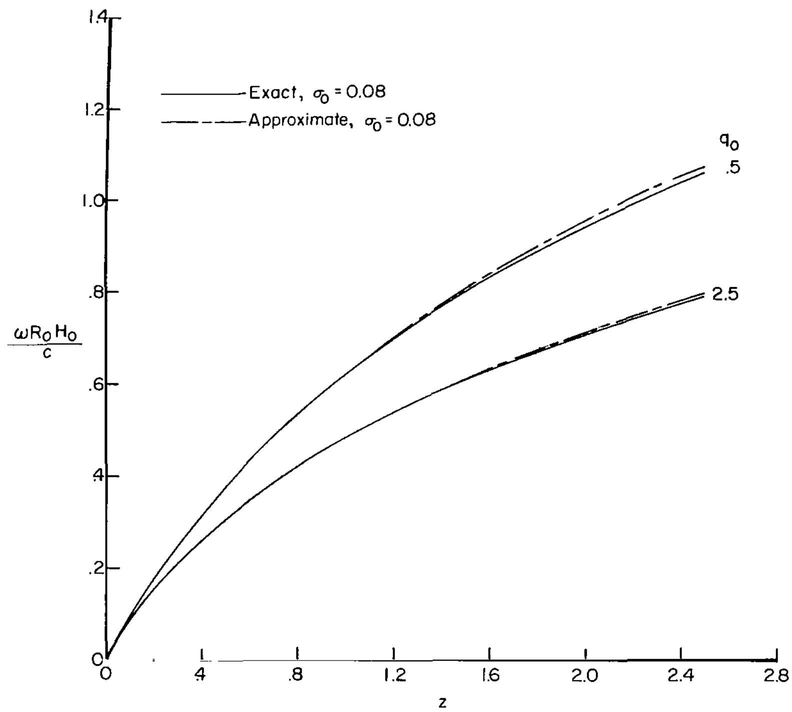


Figure 7.- Differences in $\frac{\omega R_0 H_0}{c}$ for the exact and approximate finite density models of the universe. (Eqs. (23) and (39).)

The integration of equation (39) for ω gives

$$\omega = \frac{1}{\sqrt{-k}} \sqrt{\frac{1 + q_0 - 3\sigma_0}{1 + q_0 + 3\sigma_0}} \left[\cosh^{-1} \frac{\left(1 + q_0 + 3\sigma_0\right)z + \left(q_0 + 1\right)}{\sqrt{q_0\left(q_0 + 1\right) - 3\sigma_0}} - \cosh^{-1} \frac{q_0 + 1}{\sqrt{q_0\left(q_0 + 1\right) - 3\sigma_0}}\right] \tag{40}$$

and substitution into equation (21a) for $S(\omega)$ gives

$$S(\omega) = \frac{1}{\sqrt{-k}} \sinh\left\{\sqrt{\frac{1 + q_o - 3\sigma_o}{1 + q_o + 3\sigma_o}}\left[\cosh^{-1} \frac{(1 + q_o + 3\sigma_o)z + (q_o + 1)}{\sqrt{q_o(q_o + 1) - 3\sigma_o}} - \cosh^{-1} \frac{q_o + 1}{\sqrt{q_o(q_o + 1) - 3\sigma_o}}\right]\right\} \tag{41}$$

Figures 8 and 9 compare the ω and $S(\omega)$ for the exact and approximate models. The exact solutions for ω and $S(\omega)$ are given by equations (23) and (21a), and the approximate solutions by equations (40) and (41). The differences in ω and $S(\omega)$ for the exact and approximate models are small, on the order of those found for $\omega H_o R_o/c$.

As indicated in figures 7 to 9, this approximate solution for ω and $S(\omega)$ has been checked for $\sigma_o = 0.08$, and $q_o = 0.5$ and 2.5. Charts are given in appendix A that show the σ_o and q_o combinations that give a 2- and 4-percent error in this approximation for ω.

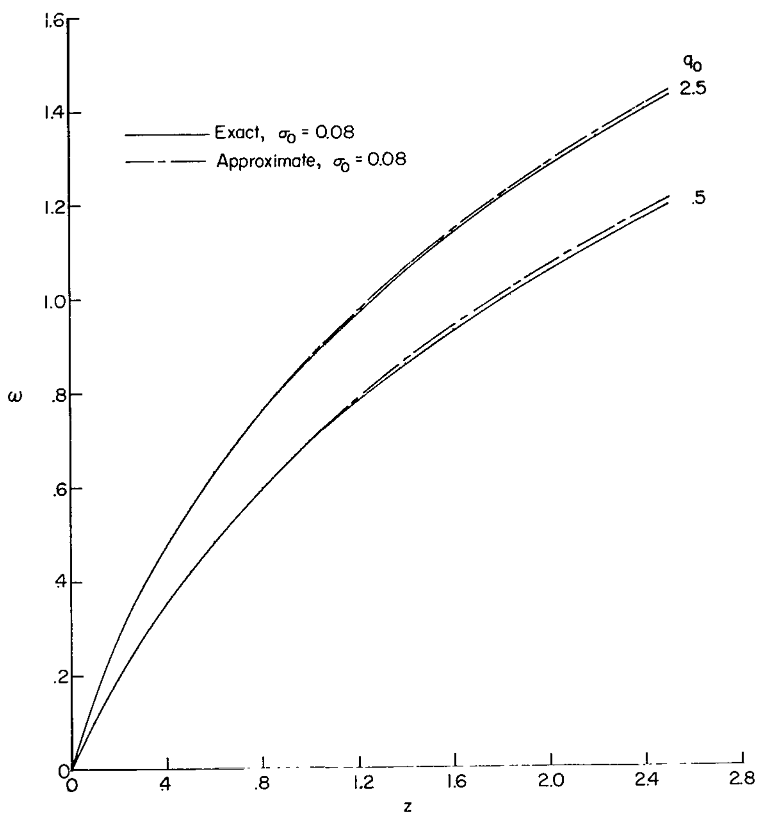


Figure 8.- Differences in ω for the exact and approximate finite density models of the universe. (Eqs. (23) and (40).)

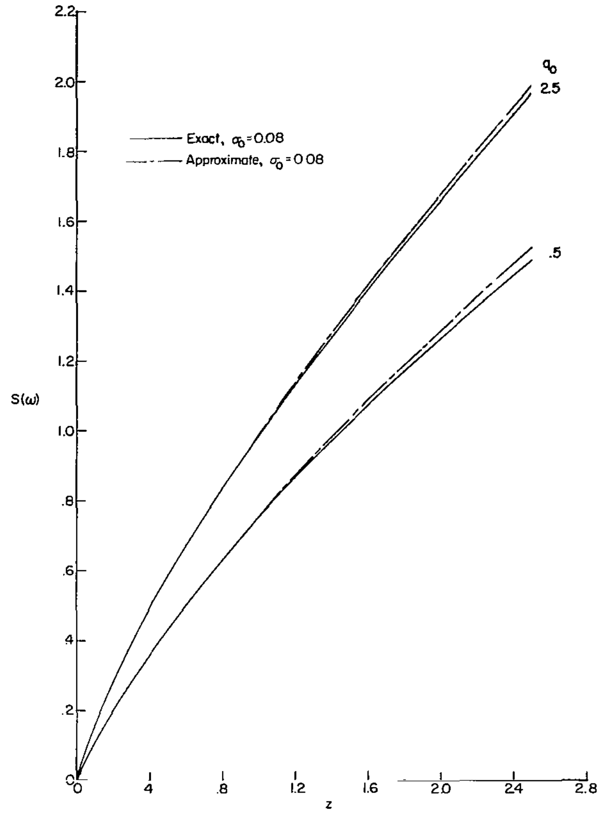


Figure 9.- Differences in S(ω) for the exact and approximate finite density models of the universe. (Eqs. (21a), (23), and (41).)

Another method to obtain an approximation for ω is to expand the integral in a McLaurin series about $z = 0$ and then expand the expression for $S(\omega)$ in an appropriate series. This form of the red-shift—magnitude relation, called the expansion form, has been extensively studied (see ref. 15), and the red-shift—magnitude relation using this approach is

$$m = 5 \log_{10} z\left(1 + \frac{1 - q_0}{2} z\right) + C \tag{42}$$

This relation is restricted by assumptions made in its development to small values of z, that is, $z \ll 1$. Sherman (ref. 1) and Mattig (ref. 22) have compared this form of the red-shift—magnitude relation with the zero-density and $\Lambda = 0$ red-shift—magnitude relations and found differences of 1.8 magnitude and 0.4 magnitude, respectively. As the zero-density model and the $\Lambda = 0$ model are special solutions of the exact case, there is no reason to expect better agreement between the exact and the expansion model than has been demonstrated in the special cases.

The Scale Factor and Time Equations

There is other information that can be obtained from the model universe. This information is the scale factor, time of light travel, and time since the beginning of expansion. Equation (7b) is integrated to obtain either the time of light travel or the time since the beginning of expansion.

The starting point for these calculations is equation (10) which is equation (7b) in integral form. Equation (10) in the form

$$t_o - t = H_o^{-1}\int_0^z \frac{dz}{(1+z)\left[2\sigma_o z^3 + \left(3\sigma_o + q_o + 1\right)z^2 + 2\left(q_o + 1\right)z + 1\right]^{1/2}} \tag{43}$$

is used to determine the time of light travel and in the form

$$T = H_o^{-1}\int_0^z \frac{dy}{\left[\left(\sigma_o - q_o\right)y^2 + \left(1 + q_o - 3\sigma_o\right) + \frac{2\sigma_o}{y}\right]^{1/2}} \tag{44}$$

is used to determine the time since the beginning of expansion. In equation (44), T is the time since the beginning of expansion and $y = \frac{R}{R_o}$.

Time of Light Travel

Equation (43) is integrated to obtain the time of light travel. The solution to equation (43) is of the form $t_o - t = f(R)$. This solution is then inverted to give the scale factor $R = h\left(t_o - t\right)$.

<u>The case of two equal roots</u>.- The cubic in equation (43) has equal roots for certain combinations of σ_o and q_o when $\Lambda \neq 0$ and always when $\Lambda = 0$. When $\Lambda \neq 0$ and the cubic has two equal roots, equation (43) becomes

$$t_o - t = \frac{H_o^{-1}}{\sqrt{2\sigma_o}}\int_0^z \frac{dz}{(1+z)\left(z + \frac{q_o + 1}{3\sigma_o}\right)\sqrt{z + \frac{9\sigma_o - \left(q_o - 1\right)}{6\sigma_o}}} \tag{45}$$

This integral may be written in the form

$$t_o - t = \frac{3\sigma_o H_o^{-1}}{\sqrt{2\sigma_o}\left(q_o + 1 - 3\sigma_o\right)}\int_0^z \left[\frac{1}{(z+1)\sqrt{z + \frac{9\sigma_o - \left(q_o + 1\right)}{6\sigma_o}}} - \frac{1}{\left(z + \frac{q_o + 1}{3\sigma_o}\right)\sqrt{z + \frac{9\sigma_o - \left(q_o + 1\right)}{6\sigma_o}}}\right]dz \tag{46}$$

and in this form the equation for $t_o - t$ can be easily integrated by standard formulas. The integrals in equation (45) or (46) give $t_o - t$ only when σ_o and q_o satisfy the condition $\left[3\sigma_o - \left(q_o + 1\right)\right]^3 - 27\sigma_o^2\left(\sigma_o - q_o\right) = 0$. Figure 2 shows a plot of the values of q_o and σ_o that satisfy this condition. In the case of $t_o - t$, the expression resulting from the integration of equation (46) is not as simple as it was for ω and, in general, will consist of two inverse tangent functions, or two logarithmic functions, or a combination of inverse tangent and logarithmic functions. There are 11 possible combinations of these functions and the specific combination depends on the σ_o and q_o used in the integration of equation (46). Because of the many different conditions that would have to be covered and the highly restricted nature of the solutions, the integration of equation (46) is not presented in detail. However, it is easily integrated by using the standard formulas. The second case of two equal roots occurs when $\Lambda = 0$. When $\Lambda = 0$, two roots are equal and equation (43) becomes

$$t_o - t = \frac{H_o^{-1}}{\sqrt{2q_o}} \int_0^z \frac{dz}{(1 + z)^2 \sqrt{z + \frac{1}{2q_o}}} \tag{47}$$

and the necessary condition is that $\sigma_o = q_o$. In this case it is simple to use equation (10) with $\Lambda = 0$ to obtain

$$t_o - t = \int_R^{R_o} \frac{\sqrt{R}\, dR}{\left(2H_o^2 R_o^3 q_o - kc^2 R\right)^{1/2}} \tag{48}$$

Because of difficulties that arise when inverting the solution of equation (48) to obtain the scale factor, it is convenient to introduce the dummy variable ζ when integrating for $k = -1$ or 1. When $k = 0$, the inversion of the integration of equation (48) presents no difficulties. When $k = -1$, the substitution $\frac{c^2}{2H_o^2 R_o^3 q_o} R = \sinh^2\zeta$ is used and when $k = 1$, the substitution $\frac{c^2}{2H_o^2 R_o^3 q_o} R = \sin^2\zeta$ is used. The results of the integration of equation (48) for $k = -1$ and $k = 1$ by using the substitutions, and the direct integration when $k = 0$ are as follows:

$$\left.\begin{aligned} t_o - t &= \frac{q_o}{H_o\left(1 - 2q_o\right)^{3/2}}\left[\cosh\left(\zeta_o + \zeta\right)\sinh\left(\zeta_o - \zeta\right) + \zeta - \zeta_o\right] && (k = -1) \\ R &= \frac{cq_o}{H_o\left(1 - 2q_o\right)^{3/2}}(\cosh 2\zeta - 1) && (k = -1) \end{aligned}\right\} \tag{49}$$

$$\left.\begin{aligned} t_o - t &= \sqrt{\frac{2}{9H_o q_o}}\left(1 - \frac{R}{R_o}\right)^{3/2} && (k = 0) \\ R &= R_o\left\{1 - \left[\sqrt{\frac{9H_o q_o}{2}}\left(t_o - t\right)\right]^{2/3}\right\} && (k = 0) \end{aligned}\right\} \tag{50}$$

$$\left.\begin{aligned} t_o - t &= \frac{q_o}{H_o\left(1 - 2q_o\right)^{3/2}}\left[\zeta_o - \zeta + \sin\left(\zeta_o - \zeta\right)\cos\left(\zeta_o + \zeta\right)\right] \qquad (k = 1) \\ R &= \frac{cq_o}{H_o\left(2q_o - 1\right)^{3/2}}(1 - \cos 2\zeta) \qquad (k = 1) \end{aligned}\right\} \tag{51}$$

The foregoing equations for $t_o - t$ and R are not too useful in that a knowledge of one of the variables is required in order to determine the other. In the case of R for $k = \pm 1$, the equations are transcendental as ζ is a function of R. However, all the foregoing equations may be expressed as a function of the red shift and acceleration parameters through the use of the relation $\frac{R_o}{R} = 1 + z$ and equation (8). When using equation (8) for the $\Lambda = 0$ case, σ_o is set equal to q_o.

The zero-density model.- In the case of the zero-density model universe, equation (10) becomes

$$t_o - t = H_o^{-1}\int_R^{R_o}\frac{dR}{\left[R_o^{2}\left(q_o + 1\right) - q_o R^{2}\right]^{1/2}} \tag{52}$$

The integration of equation (52) gives the general expression for $t_o - t$ for the zero-density model which is

$$H_o\left(t_o - t\right) = \sqrt{q_o}\left(\cos^{-1}\frac{R}{R_o}\sqrt{\frac{q_o}{q_o + 1}} - \cos^{-1}\sqrt{\frac{q_o}{q_o + 1}}\right) \tag{53}$$

Figure 10 compares $t_o - t$ for the zero-density model universe with the $t_o - t$ for the general case which is given by equation (43). As can be seen, there is excellent agreement between the two solutions where $\sigma_o = 0.08$ and 0 and $q_o = 0.5$ and 2.5.

The scale factor R is obtained by inverting equation (53) so that R is given as a function of $t_o - t$. This inversion gives for the zero-density model

$$R = R_o\left[\cos\frac{\sqrt{q_o}\left(t_o - t\right)}{H_o^{-1}} - \frac{1}{\sqrt{q_o}}\sin\frac{\sqrt{q_o}\left(t_o - t\right)}{H_o^{-1}}\right] \tag{54a}$$

and is a general expression for R for the zero-density model universe. Figure 11 compares solutions for R for the exact and zero-density models and the agreement is poor. If a modified zero-density scale factor R^* is adopted

$$R^* = \frac{\left(R_o\right)_e}{\left(R_o\right)_d}(R) \tag{54b}$$

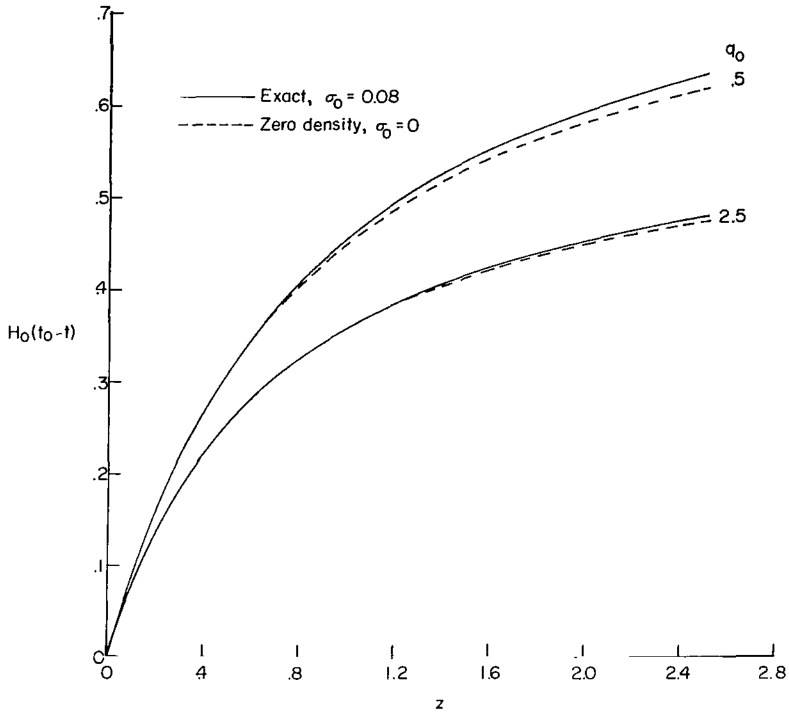


Figure 10.- Differences in $H_0(t_0 - t)$ for the exact finite density and zero-density models of the universe. (Eqs. (43) and (53).)

as was done for ω and $S(\omega)$, as shown in figure 12, there is very good agreement between the exact and zero-density modified scale factors.

The zero-density model universe and the $\Lambda = 0$ model universe have a common model and that model is for $q_0 = 0$. The restrictions on the use of ω^* in the zero-density model apply to the use of R^* when it is used to determine $t_0 - t$ and R.

Approximate solutions for $t_0 - t$ and R.- The approximate solutions for $t_0 - t$ and R make use of the same approximation that was used for obtaining the approximate solutions for ω, that is, the term $2\sigma_0 z^3$ makes a negligible contribution to the integral in equation (43). When the term $2\sigma_0 z^3$ is assumed to make a negligible contribution to the cubic in equation (43), equation (43) becomes

$$t_0 - t = H_0^{-1} \int_0^z \frac{dz}{(1 + z)\left[\left(3\sigma_0 + q_0 + 1\right)z^2 + 2\left(q_0 + 1\right)z + 1\right]^{1/2}} \tag{55}$$

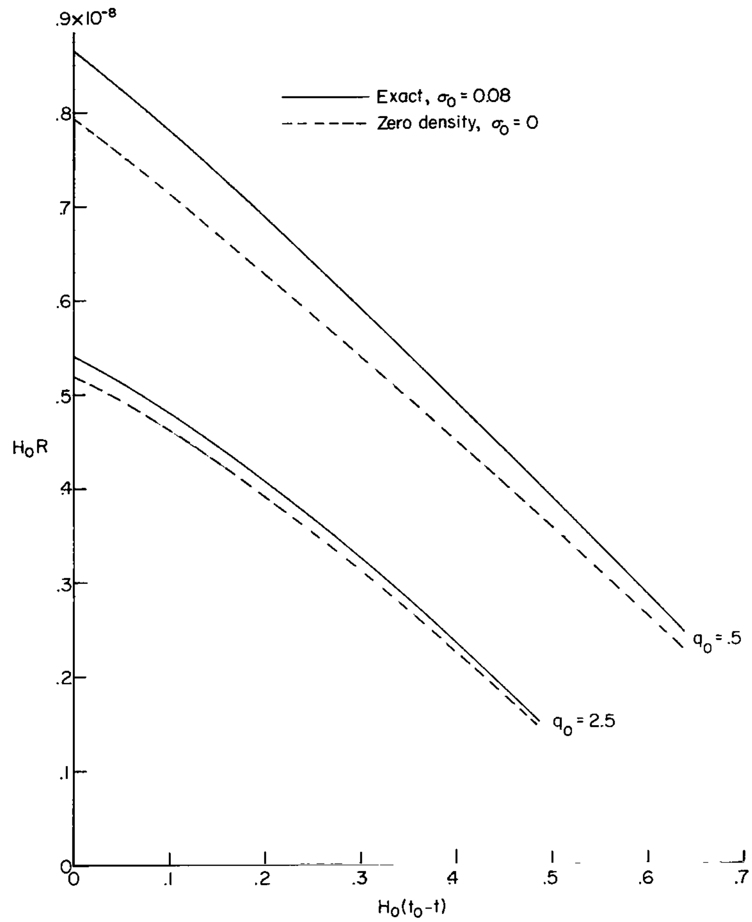


Figure 11.- Differences in H_0R for the exact finite density and zero-density models of the universe. (Eqs. (43) and (54).)

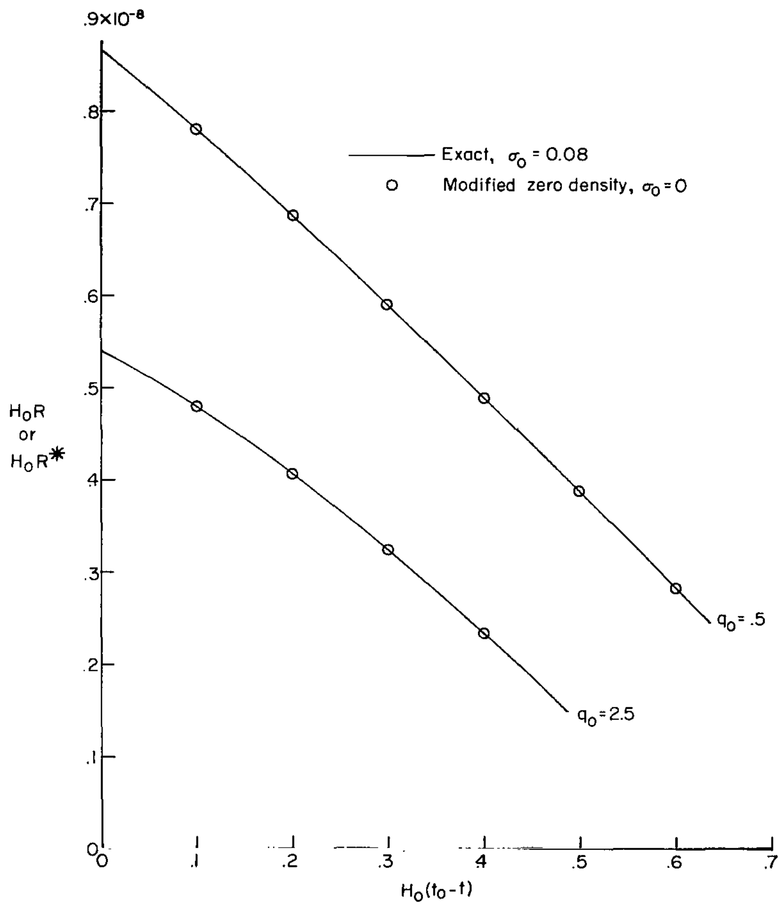


Figure 12.- Differences in H_0R or H_0R^* for the exact finite density and modified zero-density models of the universe. (Eqs. (53) and (54b).)

and integrates to

$$t_o - t = \frac{H_o^{-1}}{\sqrt{q_o - 3\sigma_o}}\left[\sin^{-1}\frac{(3\sigma_o - q_o)R - 3\sigma_o R_o}{R_o\sqrt{q_o(q_o + 1) - 3\sigma_o}} - \sin^{-1}\frac{-q_o}{\sqrt{q_o(q_o + 1) - 3\sigma_o}}\right] \tag{56}$$

As shown by figure 13, equation (56) is an excellent approximation for $t_o - t$ given by the exact expression (eq. (43)). Inversion of equation (56) gives the expression for R which is

$$R = \frac{R_o}{\sqrt{q_o - 3\sigma_o}}\left[\frac{q_o}{\sqrt{q_o - 3\sigma_o}}\cos\frac{\sqrt{q_o - 3\sigma_o}}{H_o^{-1}}(t_o - t) - \sin\frac{\sqrt{q_o - 3\sigma_o}}{H_o^{-1}}(t_o - t)\right] - \frac{3\sigma_o R_o}{q_o - 3\sigma_o} \tag{57}$$

As in the case of ω, $t_o - t$ and R obtained from this approximation are not applicable to the $\Lambda = 0$ models. For negative values of q_o, the results obtained from the approximate solution are not as good as those for $q_o > 0$. The approximation becomes worse when $q_o = -(1 + 3\sigma_o)$, and the approximation improves when q_o becomes more negative.

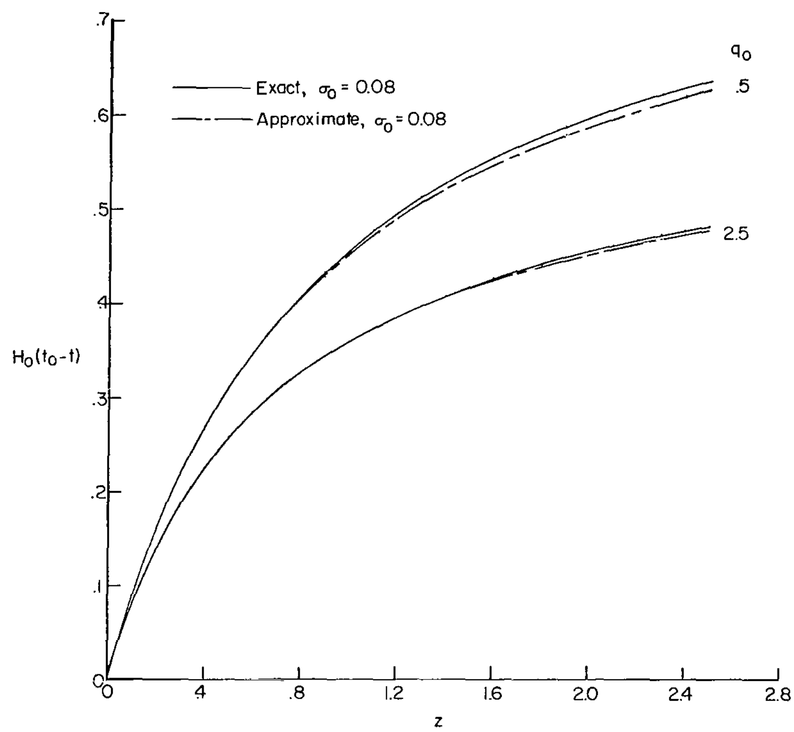


Figure 13.- Differences in $H_o(t_o - t)$ for the exact and approximate finite density models of the universe. (Eqs. (56) and (43).)

Charts that show these combinations of σ_0 and q_0 that give a difference between the exact and approximate solutions of 2 percent and less or 4 percent and less are presented in appendix A.

Time Since the Beginning of Expansion

Equation (44) gives the time since the beginning of expansion for those models of the universe that have $R = 0$ at $t = 0$ and excludes models 10, 11, and 14 of table I. If the proper initial value of R at $t = 0$ is used for the lower limit, a time since the beginning of expansion can be calculated for models 11 and 14. In the case of model 10, time since the beginning of expansion is meaningless because this model is a static model and there is no expansion.

Equation (44) is a definite integral, and it can be easily integrated for the zero-density model and the $\Lambda = 0$ model, the significant family of models when two roots are equal. In the case of the approximate model, equation (44) takes the form

$$T = H_0^{-1} \int_0^1 \frac{dy}{\left[\left(3\sigma_0 - q_0\right)y^2 - 6\sigma_0 y + \left(3\sigma_0 + q_0 + 1\right)\right]^{1/2}} \tag{58}$$

which can also be easily integrated in terms of simple functions.

Calculations of the time since the beginning of expansion are mainly useful in determining whether the time scale inequality which is

$$\text{Time since beginning of expansion} \geqq \text{Age of oldest stars} \tag{59}$$

is satisfied for a given model of the universe. Equation (59) is meaningful only when a model starts from the point $R = t = 0$. At this point it is assumed that no stars and galaxies existed and that they were formed shortly after expansion started. Thus, time since the beginning of expansion has significance in these models. However, for models 11 and 14 that have open negative time axes and finite scale factors, equation (44) loses its meaning because stars and galaxies could have existed before expansion started.

DISCUSSION

Four simple models of the universe have been derived and equations that relate observation and theory have been obtained for each model. Of the four simple models, only two are significant because fewer restrictions are incorporated. These models are the zero-density and approximate model. Both of these models allow all values of k and Λ and if the density is on the order of that predicted by observation or less $\left(\rho \approx 10^{-30}\ \mathrm{g/cm^3}\right.$ or less$\left.\right)$, then these models are highly useful. The other two models, those that have two equal roots, are not considered significant because of the restrictions placed on σ_o and q_o by the condition of two equal roots. However, one of these two models, the model with $\Lambda = 0$, has been used in the past (ref. 19) and therefore is compared with the zero-density model. One other model, the expansion model (eq. (42)), is considered in the comparison because of its frequent use.

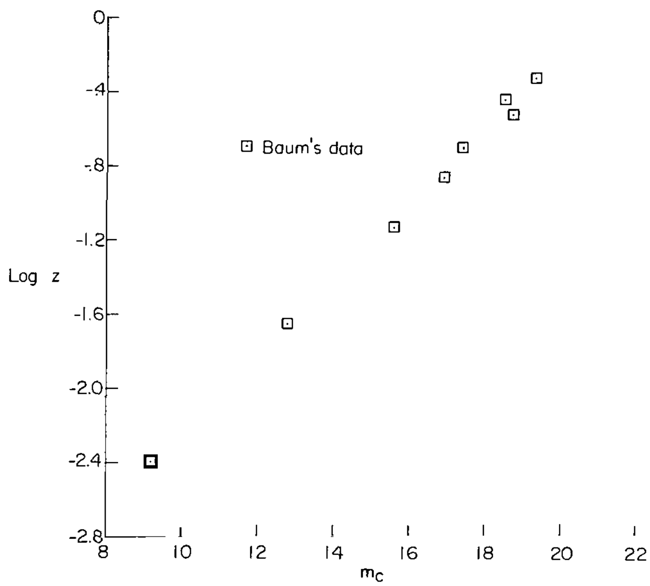


Figure 14.- Red-shift—apparent-magnitude data used for the comparison of models.

It was decided to use the red-shift—magnitude relation for the exact zero density, $\Lambda = 0$, approximate and expansion models to determine σ_o and q_o from a set of observational data. Two sets of observational data are available: one by Humason, Mayall, and Sandage for 18 clusters of galaxies and one by Baum for 8 clusters of galaxies. Both of these sets of data are summarized in reference 3. Because Baum's data (fig. 14) contain six color photoelectric magnitudes, no K correction is required. For this reason and because Baum's data appeared to have less scatter, it was decided to use it for the calculations to compare the various models of the universe presented in this paper. The results of the calculations are given in table II.

TABLE II.- RESULTS OF CALCULATIONS

Model used	Equation	σ_o	q_o	k	Λ	Model (Table I)
Exact	(14), (21a), and (23)	3.029	0.0077	1	>0	9 (Le Maitre)
Approximate	(14) and (41)	3.20	-.0059	1	>0	9 (Le Maitre)
$\Lambda = 0$	$(14)_{\sigma_o=q_o}$, (21a), (30), (31), and (32)	1.56	1.56	1	0	8 (Oscillating)
Zero density	$(14)_{\sigma_o=0}$ and (36)	0	2.934	-1	<0	1 (Oscillating)
Expansion	(42)	----	1.39	-1*	<0*	1 (Oscillating)

*Equations (8) and (9) were used to compute k, and Λ for this model at a value of 0.45 $\left(\rho_o \approx 10^{-30}\ \text{g/cm}^3\right)$ was used.

The results presented in table II are the initial results of a study of the problem of determining σ_o and q_o from observational data and are results for a very limited amount of data. The model characteristics that are indicated when density is neglected (expansion model) or a restrictive assumption made ($\Lambda = 0$ and zero-density models) disagree with the model characteristics that are indicated when the exact red-shift—magnitude relation is used. However, the σ_o and q_o predicted by the approximate model are close to those predicted by the exact model. This result is not surprising since the σ_o and q_o obtained through the use of the exact model lie in a region where the approximate model is a good approximation for the exact model if an error between 2 and 4 percent in the elliptic integral is tolerable. Both the exact and approximate models predict σ_o on the order of 3.0; this prediction represents a density that is about 20 times higher than the maximum density based on observation obtained by Oort (ref. 13) and about 60 times Abell's best estimate (ref. 12). If the trends established by these initial results are substantiated by further calculations with more extensive data, only the exact and approximate models should be used for the analysis of observational data. Solheim's results (ref. 23) appear to confirm these results.

The results presented in table II have been plotted in figure 15 with the data superimposed on the curves. The most striking fact about this figure is that for the range of z covered by data, 0.004 to 0.460 (that is, $\log z = -2.398$ to $\log z = 0.361$), all the

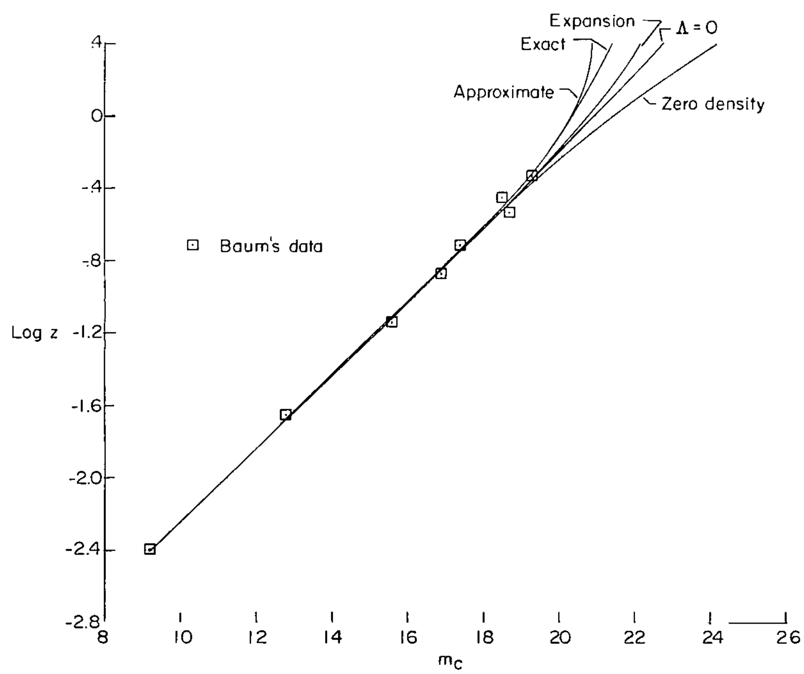


Figure 15.- Comparison of several model universes with red shift z and apparent magnitude m_c for Baum's eight clusters from reference 3.

models of table II appear to fit the data equally well. Above $z = 0.5$ ($\log z = -0.30103$), the various models diverge. Present data for galaxies do not extend above $z = 0.5$, and data with $z > 0.5$, based on the evidence of figure 15, are required to determine a model of the universe.

Quasi-stellar radio sources (see ref. 24) have measured red shifts greater than 0.5; however, these sources were not used to extend the data for galaxies used in the calculations because (1) the cause of the red shift is in doubt and (2) the intrinsic luminosity of these sources appears to vary widely and is different from that of the galaxies. The use of the red-shift—magnitude relation requires the assumption that all sources have about the same intrinsic luminosity. Additional data from quasi-stellar radio sources published since reference 24 have not clarified the situation.

CONCLUDING REMARKS

Under the assumption that general relativity is the theory of gravitation that best represents the observed universe, relativistic models that are both isotropic and homogeneous have been studied. In addition to the assumptions of isotropy and homogeneity, it was necessary to assume that pressure terms could be neglected with respect to density terms in order to solve the field equations that describe the universe.

The solutions for the scale factor and radial metric variable ω, which are generally elliptic integrals, have been studied for the exact case and several simplified cases. In one of the simplified cases, the integrals for the scale factor and radial metric variable have two equal roots and although both integrals can be integrated by simple functions, the condition of two equal roots imposes too severe a restriction on the possible values of the density and acceleration parameters for the solutions with two equal roots to be of more than academic interest. Models with the cosmical constant equal to zero are contained in the case of two equal roots and the same remarks apply. In addition, the cosmical constant is a constant of integration in the field equations and should not be arbitrarily set to zero. Simplified models that have two equal roots should not be used unless the analysis of observational data with a more general model indicates that the special conditions for the integrals for the scale factor and radial metric variable to have two equal roots exist.

Zero-density model universes and an approximate model in which the cubic term was assumed to make a negligible contribution to the integral in the definition of the radial metric variable ω were also studied to determine their applicability to the analysis of observational data.

A computing program based on the method of differential corrections and the data for Baum's eight clusters of galaxies was used to evaluate the exact, approximate, $\Lambda = 0$, zero-density, and expansion models. Both the exact and approximate models predict a high-density universe and the acceleration parameter for these models is not in agreement with that determined by the $\Lambda = 0$, zero-density, and expansion models. Because of the high density and the lack of agreement between the values of the acceleration parameter determined by the various models, except in the case of the exact and approximate models which predict an almost zero value for the acceleration parameter, it was concluded that only the exact and approximate models should be used for the analysis of observational data. The approximate model should be used only if an error between 2 and 4 percent in the integral in the radial metric variable ω is considered tolerable.

Further study of the results obtained from the computing program showed that in the range of red shift covered by present data, all the models used appeared to fit the data equally well, and data that include red shifts greater than one-half are required to define a model of the observed universe.

Although the data for quasi-stellar radio sources fall in the required range of red shift, it was pointed out that, as yet, these data are not considered in the problem of determining a model of the universe because of the uncertainty as to the cause of the red shift and because the intrinsic luminosity of quasi-stellar radio sources shows a wide variation and is different from that of galaxies.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., September 21, 1967,
188-41-01-01-23.

APPENDIX A

DETERMINATION OF REGIONS OF APPLICABILITY OF THE APPROXIMATE MODEL

The region of σ_O, q_O over which the approximate model is applicable is determined for errors of 2 and 4 percent introduced through use of the approximate model. The only difference between the approximate and exact models is the approximation of equation (23) by equation (39) for ω and the approximation of equation (48) by equation (55). These equations were set up as inequalities, and a high-speed digital computer was used to determine σ_O and q_O as a function of z. The following inequalities were used:

$$\frac{I(z)_{39} - I(z)_{23}}{I(z)_{23}} \leqq \epsilon \tag{A1}$$

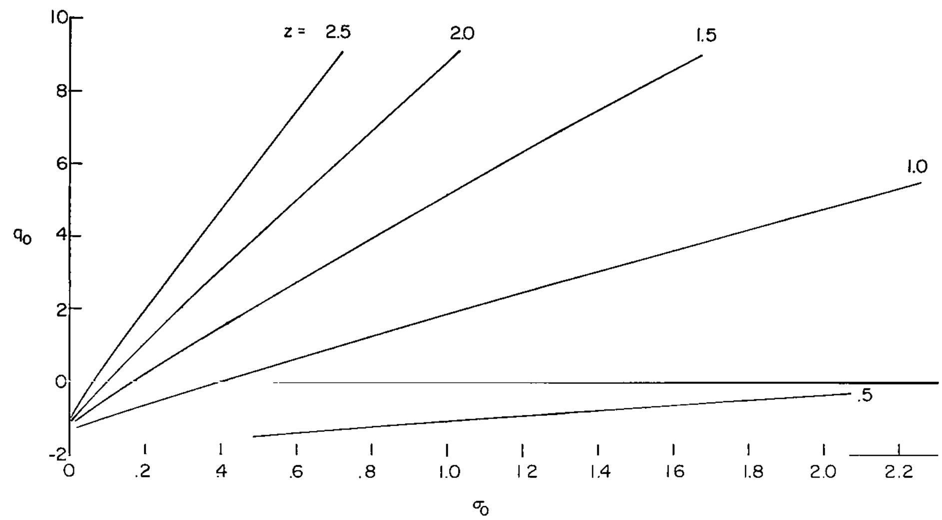
for ω and

$$\frac{I(z)_{55} - I(z)_{48}}{I(z)_{48}} \leqq \epsilon \tag{A2}$$

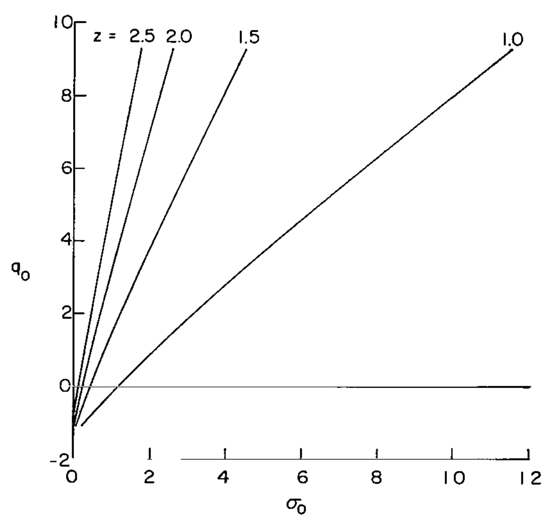
for $t_O - t$. In these equations, I(z) indicates the integral and the subscript denotes the equations from which it came. Two values of the error ϵ, 0.02 and 0.04, were used.

The results of the calculations are presented in figure 16 for ω and in figure 17 for $t_O - t$. When ϵ was 4 percent, it was found that the error for $z = 0.5$ was always less than 4 percent for the range of σ_O and q_O considered; thus, no boundary for $z = 0.5$ is shown in figures 16 and 17, the region between the axis and the lines of constant z is usable, and a combination of σ_O and q_O that falls on or inside of a line of constant z is usable. For instance, if σ_O and q_O fall on the $z = 1.0$ line, the approximate model could be used to determine σ_O and q_O out to $z = 1.0$. The maximum error would occur on the $z = 1.0$ line; if σ_O and q_O fall inside this line, the error is less.

In practice, the charts would be used as follows. The approximate model would be used to determine σ_O and q_O, and then the determined σ_O and q_O and the maximum z used would be entered into the chart to determine whether the approximation is valid.

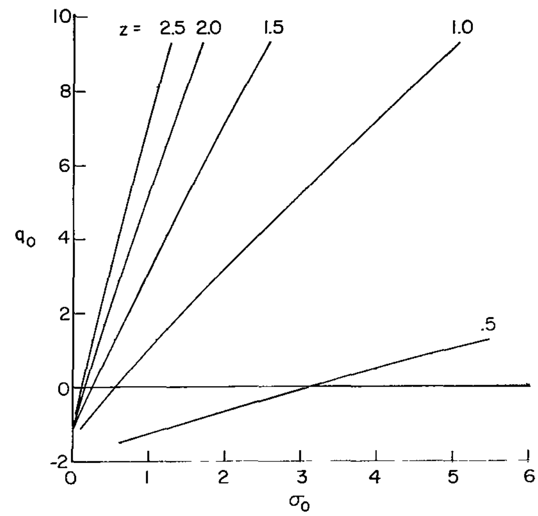


(a) $\varepsilon = 0.02$.

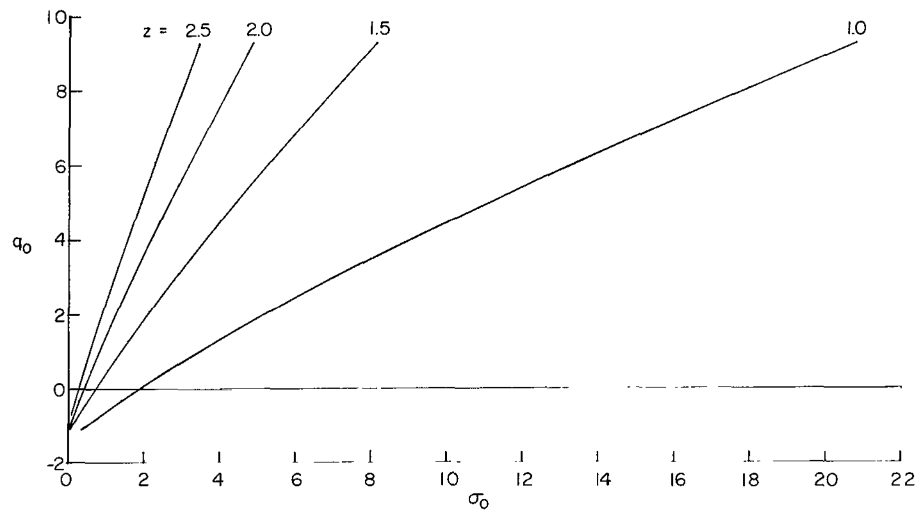


(b) $\varepsilon = 0.04$.

Figure 16.- Regions in which the approximate models give valid solutions for ω. The useful region is between the q_0 axis and lines of constant z. (Eq. (A1).)



(a) $\varepsilon = 0.02$.



(b) $\varepsilon = 0.04$.

Figure 17.- Regions in which the approximate model gives valid solutions for $t_0 - t$. The useful region is between the q_0 axis and lines of constant z. (Eq. (A2).)

APPENDIX B

SYMBOLS

$a = 7.564 \times 10^{-15}$

a,b	roots for cubic with two equal roots

$C = 5 \log_{10} \frac{c}{H_0} + M_0 - 5$

c	speed of light in vacuo
D_c	distance by measuring rod, $R_0 S(\omega)$
D_l	luminosity distance
E	curvature scalar
$E_{\mu\nu}$	contracted Riemann-Christoffel tensor
f( )	function
G	universal constant of gravitation
$g_{\mu\nu}$	metrical tensor
H_0	Hubble parameter
h	inverted function
I(z)	integral
K	red-shift correction
k	curvature constant
M_0	absolute magnitude of equivalent local source
m	apparent magnitude
m_c	corrected apparent magnitude

APPENDIX B

$N(m)$	number of like sources brighter than a given magnitude
n	number of like sources per unit volume
p	total pressure
Q	number of square degrees in celestial sphere
q_o	acceleration parameter
R	scale factor that describes geometric history of universe
R_1, R_2	roots of cubic in equation (10)
R_e	value of R associated with Λ_e
R^*	value of R for modified zero-density model
r	coordinate in metric subspace
$S(\omega)$	function of ω that depends on curvature of space
$S^*(\omega)$	value of $S(\omega)$ for modified zero-density model
s	time along line element
T	radiation temperature, $^{\circ}$K; also time since beginning of expansion
t	time
t_o	present time
$V_\mu = \frac{dx^\mu}{ds}$	
$V_\nu = \frac{dx^\nu}{ds}$	
v_o	random radial velocity
x^μ, x^ν	coordinates
$y = \frac{R}{R_o}$	

APPENDIX B

z	red shift
α, β, η	dummy roots of cubic
ϵ	error
ζ	linear diameter; also dummy variable
θ	coordinate in metric subspace
$\kappa = \frac{8\pi G}{c^2}$	
Λ	cosmical constant
Λ_e	critical value of cosmical constant
ξ	function of radial metric variable r
ρ	material density
ρ_o	present material density
σ_o	density parameter
ϕ	angular diameter; also coordinate in metric subspace
ω	function of radial metric variable
ω^*	value of ω for modified zero-density model

Subscripts:

e,d	exact and zero-density models
o	present values
μ, ν	tensor indices (0, 1, 2, and 3)

Dots over symbols denote differentiation with respect to time.

REFERENCES

1. Sherman, Windsor L.: Zero-Density Cosmological Models and Their Applicability to the Observed Universe. NASA TN D-2601, 1965.
2. Sherman, Windsor L.: An Approximate Solution for the Luminosity Distance in Zero-Pressure Relativistic Model Universes That Have the U-Property. NASA TN D-3047, 1965.
3. McVittie, G. C.: General Relativity and Cosmology. Univ. of Illinois Press, 1965.
4. Einstein A.: Cosmological Considerations on the General Theory of Relativity. The Principle of Relativity, W. Perrett and G. B. Jeffery, trans., Dover Publ., Inc., 1923, pp. 177-188.
5. North, J. D.: The Measure of the Universe. Clarendon Press, c.1965.
6. Einstein, A.; and de Sitter, W.: On the Relation Between the Expansion and the Mean Density of the Universe. Proc. Natl. Acad. Sci., U.S., vol. 18, no. 3, Mar. 15, 1932, pp. 1-214.
7. Einstein, Albert: The Meaning of Relativity. Princeton Univ. Press, 1946.
8. Einstein, A.: Do Gravitational Fields Play an Essential Part in the Structure of the Elementary Particles of Matter? The Principle of Relativity, W. Perrett and G. B. Jeffery, trans., Dover Publ., Inc., 1923, pp. 191-198.
9. Weyl, Hermann (Henry L. Brose, trans.): Space – Time – Matter. Dover Publ., Inc., 1922.
10. McVittie, G. C.: The Cosmical Constant and the Time-Scale. J. Soc. Ind. Appl. Math., vol. 10, no. 4, Dec. 1962, pp. 756-767.
11. Heckmann, O.: General Review of Cosmological Theories. Problems of Extra-Galactic Research, G. C. McVittie, ed., The MacMillan Co., c.1962, pp. 429-440.
12. Abell, G. O.: Clustering of Galaxies. Annual Review of Astronomy and Astrophysics, vol. 3, Leo Goldberg, Armin J. Deutsch, and David Layzer, eds., Ann. Rev., Inc., 1965, pp. 1-21.
13. Oort, J. H.: Distribution of Galaxies and the Density in the Universe. La Structure et L'Évolution de L'Univers, R. Stoops, ed., Inst. Intern. Phys. Solvay, 1958, pp. 163-183.
14. Sandage, Allan: The Ability of the 200-Inch Telescope To Discriminate Between Selected World Models. Astrophys. J., vol. 133, no. 2, Mar. 1961, pp. 355-392.
15. Allen, C. W.: Astrophysical Quantities. Second ed., The Athlone Press (London), 1963.

16. Anon.: The Black-Body Universe – Spectrum Extended. Research Facilities and Programs. Physics Today, vol. 19, no. 9, Sept. 1966, pp. 75-76.

17. Robertson, H. P.: Relativistic Cosmology. Rev. Mod. Phys., vol. 5, Jan. 1933, pp. 62-90.

18. Bondi, H.: Cosmology. Second ed., Cambridge Univ. Press, 1960.

19. Robertson, H. P.: The Theoretical Aspects of the Nebular Redshift. Publ. Astron. Soc. Pacific, vol. 67, Apr. 1955, pp. 82-98.

20. Mattig, W.: Über den Zusammenhang zwischen der Anzahl der extragalaktischen Objekte und der scheinbaren Helligkeit. Astron. Nachr., vol. 285, 1959, pp. 1-2.

21. Sandage, Allan: Observational Tests of World Models. J. Soc. Ind. Appl. Math., vol. 10, no. 4, Dec. 1962, pp. 781-794.

22. Mattig, W.: Über den Zusammenhang zwischen Rotverschiebung und scheinbarer Helligkeit. Astron. Nachr., vol. 284, 1958, pp. 109-111.

23. Solheim, J. E.: Relativistic World Models and Redshift-Magnitude Observations. Monthly Notices, Royal Astron. Soc., vol. 133, no. 3, 1966, pp. 24, 321-338.

24. Sandage, Allan: The Existence of a Major New Constituent of the Universe: The Quasi-Stellar Galaxies. Astrophys. J., vol. 141, no. 4, May 15, 1965, pp. 1560-1578.

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